Sharing information: Could Experts Consolidate?

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Abstract

In this paper, we analyze a cooperative model of information sharing among experts under four types of information structure, three of which are standard assumptions in the literature. We construct a transferable utility game, called *commission games*, which captures the value of information for a coalition of experts. We find that the core is empty for commission games that have information structures that satisfy symmetric monotone likelihood ratio property, conditional independence, or perfect correlation. We find a necessary condition that a weaker form of monotone likelihood ratio property leads to indifference between sharing and no sharing if the core exists. Lastly, we give a sufficient condition on the information structure for existence of core, which imposes strong complementarity of information between experts.

1 Introduction

Information aggregation in the literature has mostly been discussed in the context of voting problems. It may be under the context of co-aligned voters like in the classical Condorcet Jury framework or with strategic voters (Feddersen and Pesendorfer, 1996, 1998), or differently aligned voters (Levy and Razin, 2015; Ortoleva and Snowberg, 2015). Agents are endowed with signals, correlated or otherwise, and vote on an outcome that best fits their posterior belief and agenda. Then, the group of agents aggregate their choices and implement a public decision that would affect each agent's payoff individually. This means that even if the decision was taken together each agent internalizes the outcome, implying that the social surplus generated from combining private information is simply the sum of individual payoff of the agents.

However, this does not capture public deliberation where information can be rationally shared among peers before making a decision or recommendation. As an example, suppose that there is a set of agents where each one has private information. If two agents can generate surplus from their combined information more than their individual private information, then they would be willing to agree and share their respective information with each other. This illustrates the concept of individual rationality for sharing information. Suppose that we introduce a third agent. The two agree to share their information with the third agent if the split that they get from the total surplus generated by combining everyone's information is larger than the surplus from the pair's information. This captures the idea of coalitional rationality. Hence, these privately informed agents can rationally agree to share their information among themselves so long as they can generate enough surplus for everyone in the group.

In this paper, we discuss a cooperative game theory model of information sharing and give conditions on the information structure whether it is feasible to do so. We refer to *information sharing* as a rational cooperative behavior of an agent to share their private information with the other agents as an attempt to have a better group decision. In particular, we model information sharing in the context where an agent or a group of agents have a claim over the surplus generated by their information. We employ a cooperative game framework to describe these claims over surpluses for all possible coalitions of agents. Hence, information is shared when experts are given an allocation at least as much as these claims. One assumption we take is that a coalition of agents can only prescribe one action such that the surplus generated comes from the pure informational advantage of grouping together.

We explore several classes of information structures that are entrenched in the literature, namely, information structures that satisfy (1) symmetric monotone likelihood ratio property, (2) conditional independence, or (3) perfect correlation. We find that all three classes imply that information is not shared. One strong insight that we get from these result is the fact that more information is better but only marginally for these classes of information structure. That is, an agent's marginal contribution to the value of information to a group of size T is weakly smaller than to a group of a size smaller than T. Note that information can potentially be shared for a weaker notion of monotone likelihood ratio property. Under this information structure, agents are indifferent between sharing and keeping their information private if the core exists. We provide a sufficient condition where information is shared which imposes strong complementarity. Lastly we discuss possible directions to characterize the information structure for the core to exist.

Several papers have documented information sharing in a multitude of settings. Under cooperative game theory, Slikker et al. (2000) introduces a game on information sharing. This game values the worth of a coalition of agents as the sum of each agents' utility given the optimal action for their shared information. Our game captures a simpler concept as the transferable game we describe is the value of information akin to voting models. In a similar strand, Habis and Herings (2011) and Routledge (2012, 2014) present different notions of core allocation for transferable utility games with uncertainty. They explore games where cooperation can be agreed upon in the ex-ante stage and maintained ex-post . Their contributions are related to unenforceable ex-ante agreements and the type of core allocations that can implemented. Although we keep the assumption that ex-ante agreements are enforceable, our contribution here is to provide a game with uncertainty based solely on information structures where their notions of core can be explored.

In a strategic setting, Raith (1996) models information sharing in a generalized oligopolistic framework where firms are endowed with private information regarding market characteristics and can choose whether or not to reveal their private information to other firms. Complete information sharing is a dominant strategy when actions are strategic complements (Bertrand Competition with uncertain demand), while no information sharing is a dominant strategy under strategic substitutes (Cournot Competition). Our model relates to this in a sense the all agents share their information when signals are complementary but keep them private when they are substitutable. Another strategic model of information sharing is Ottaviani and Sørensen (2001). Their model is very similar to ours in the sense that they consider a binary state-binary signal framework where a group of experts with heterogeneous accuracy make their recommendation known to the decision maker who then takes an action conditional on the information received by the experts. Furthermore, each recommendation is public in the sense that all other experts also know the recommendation. In their setting however, experts provide their recommendations sequentially and act in order to maintain/improve their reputation among their peers. Reputation is captured by the common belief of an expert i being a "good" type and is updated conditional on the expert's recommendation and the realization of the state of the world. Our setting departs from this in the sense that we focus on a cooperative framework in which no expert misreports their information. Furthermore, the experts in our model are purely motivated by the commission they receive which is dependent on the shared payoff received by the Decision Maker from choosing the right action.

The rest of the paper is structured as follows. Section 2 presents the model which introduces the investor's problem, experts' recommendation, the commission game, and the core. This discusses four classes of information structures, how each translates to a commission game, and the definition of the core related to these games. Section 3 presents the results related to the emptiness and non-emptiness of the core for each class of information structure. The final section discusses possible directions resulting from our propositions.

2 Model

2.1 Investors' problem

For the remainder of the paper we maintain the paradigm of experts and investors in our exposition. We consider a setup where there is a pool of investors where each one chooses to buy either asset a or b, but not both. Hence, their action σ is in the binary action space $\{a, b\}$. The state of world ω is also binary where $\omega \in \{A, B\}$. We assume that there is a common prior belief of state that is uniform, i.e., the probability of state being A is $\frac{1}{2}$. The payoff to the investor, given the action she takes, is contingent on the state, which has von Neumann-Morgenstern payoff of $u(a, A) = u(b, B) = 1$ and $u(a, B) = u(b, A) = 0$. This says that an investor gets something if her action matches the state and nothing otherwise.

2.2 Experts' recommendation

Each investor can potentially hire a coalition of experts T from a grand coalition of experts $N = \{1, 2, ..., n\}$. We assume that each expert can only be hired at most once so that the investor can only benefit from the experts she hires. Each expert has private information about the true state of the world that takes the form of a binary signal $s_i \in \{a, b\}$. They also share the same uniform prior belief over the state $\omega \in \{A, B\}$. The vector of signals $s = (s_1, s_2, ..., s_n) \in S$ is drawn from the conditional distribution F_ω. The unconditional probability of signal vector s is $F(s) = \frac{1}{2}F_A(s|A) + \frac{1}{2}F_B(s|B)$. Note that we allow for correlated as well as conditionally independent signals.

For a given coalition of experts T, we denote their vector of signal as s_T where $s_T = (\times_{i \in T} s_i) \in S|_T$ with a corresponding conditional probability $F_\omega(s_T) \equiv \sum_{s' \in S_{N-T}} F_\omega(s = s' \times s_T)$. We let $q(s_T)$ denote the posterior probability of state A after observing signal s_T; and so the posterior probability of state B is $1 - q(s_T)$. From Bayes' Theorem, $q(s_T)$ is given by

$$q(s_T) = \frac{\frac{1}{2}F_A(s_T)}{\frac{1}{2}F_A(s_T) + \frac{1}{2}F_B(s_T)}.$$

After observing their signal s_T, the coalition of experts recommends an action $\sigma_T \in \{A, B\}$ for the investor to take according to their posterior belief. The optimal recommendation rule for the coalition T given $q(s_T)$ is to recommend A if $q(s_T) > \frac{1}{2}$, to recommend B if $q(s_T) < \frac{1}{2}$, and recommend the state that has the most signal realizations if $q(s_T) = \frac{1}{2}$.

In the following, we focus on information structures that maintain symmetry in informativeness among individuals and coalitions of experts.[1] Consider Table 1 and the a signal vector $s = aab$. The first a corresponds to expert 1's signal, second a to expert 2, and signal b to expert 3. The mirror image of this signal vector is $\neg s = bba$. Now, the first notion of symmetry relates to the probability of a signal vector s and its mirror image $\neg s$, which says that for any $s \in S$ the probability of s given A is equal to the probability of its mirror image $\neg s$ given B. In Table 1, we have $F_A(aab) = F_B(bba) = 0.15$. This is also the case for signal vectors aaa with $F_A(aaa) = F_B(bbb) = 0.20$, bba with $F_A(bba) = F_B(aab) = 0.10$, bbb with $F_A(bbb) = F_B(aaa) = 0.05$, and so on, to all possible signal vectors.

The second notion of symmetry relates to the symmetry between experts. It states that a signal vector and all it possible permutations should have the same conditional probability. Consider signal vectors aab, aba, and baa in Table 1. Conditional on state being A, each one of the signal vectors has a probability 0.15. This is the case for signal vectors bba, bab, and abb with probability of 0.10 conditional on state being A. This condition implies that identity of the expert does not matter, that is, the induced information structure of expert 1 is the same for experts 2 and 3. We can see this in the induced 1-person information structure in Table 1(b). Furthermore, the coalition of

[1]The results below assume that information is symmetric. Note that we relax this symmetry assumption on the conditionally independent information structure. That is, we allow each expert to have different levels of accuracy with regard to their signal

experts 1 and 2 is exactly as informed as replacing 1 or 2 with expert 3. Hence, any coalition that is equal in size has the same informativeness. Overall, we combine these two notions of symmetry in informativeness which we call *anonymous information structure* as given below.

Definition 1 (Anonymous Information Structure)**.** *An information structure* (S, F) *is called an anonymous information structure if for any* $s \in S$ *we have*

$$F_\omega(s) = F_{\neg\omega}(\neg s)$$

where $\neg s = \times_i(\neg s_i)$*; and for any permutation of* s $\pi(s)$ *we have*

$$F_\omega(s) = F_\omega(\pi(s)).$$

Example 1. (Anonymous Information Structure) Table 1 shows the feature of an anonymous information structure for the case of three experts.

	aaa	aab	aba	baa	abb	bab	bba	bbb
A	0.20	0.15	0.15	0.15	0.10	0.10	0.10	0.05
B	0.05	0.10	0.10	0.10	0.15	0.15	0.15	0.20

(a) 3-person Anonymous Information Structure

	a	b
A	0.60	0.40
B	0.40	0.60

(b) Induced 1-person Information Structure

Table 1: Anonymous Information Structure

We introduce four types of information structures, three of which are based on standard assumptions on information structures in the literature. These standard assumptions are *perfectly correlated, conditionally independent,* and *monotonic information structures*. The fourth type of information structures, *complementary information structures*, are non-standard. We focus on these in detail.

2.2.1 Perfectly Correlated Information Structures

Definition 2 (Perfectly Correlated Information Structure)**.** *An information structure* (S, F) *is called perfectly correlated if the support of F is the set* $\{(a, ..., a), (b, ..., b)\}$.

A perfectly correlated information structure describes a scenario where every expert gets exactly the same signals as other experts. For example, experts that follow the same news cycle, have the same set of peers, read the same articles, and assess information similarly will have signals that are exactly match each other with the same accuracy. In the *interim* scenario where each expert knows their signal but not others' signal, she can correctly guess that the other experts received the same signal. *Ex-ante*, every expert can expect to receive the same signal regardless of the state of the world. In the succeeding discussion we assume that the signals are informative, i.e., $F_A(a, ..., a)$ and $F_B(b, ..., b)$ are both greater than 0.5.

Example 2. (Perfectly Correlated Information Structure) Table 2 shows the feature of a perfectly correlated information structure for the case of three experts. Notice that the support for both the conditional distribution is the set of signal vectors $\{aaa, bbb\}$. Furthermore, the support of the space of signal vectors for a coalition of two experts is $\{aa, bb\}$; and, the support for singleton coalition is $\{a, b\}$. Lastly, the probability over the support is maintained when the signal vector space is constrained to a subset of the grand coalition of three experts.

	aaa	aab	aba	baa	abb	bab	bba	bbb
A	0.75	0	0	0	0	0	0	0.25
B	0.25	0	0	0	0	0	0	0.75

(a) 3-person Perfectly Correlated Information Structure

	aa	ab	ba	bb
A	0.75	0	0	0.25
B	0.25	0	0	0.75

(b) Induced 2-person Information Structure

	a	b
A	0.75	0.25
B	0.25	0.75

(c) Induced 1-person Information Structure

Table 2: Perfectly Correlated Information Structure

2.2.2 Conditionally Independent Information Structures

Definition 3 (Conditionally Independent Information Structure)**.** *An information structure* (S, F) *is called conditionally independent if for any* $\omega \in \{A, B\}$ *and for all* $s \in S$

$$F_\omega(s) = \prod_{i=1}^{n} F_\omega(s(i))$$

where $F_\omega(s(i)) = \sum_{s' \in S_{N-i}} F_\omega(s = s' \times s(i))$.

Conditional independent information structure says that each expert has a signal that is conditionally independent from any other experts' signal. Here, every expert runs an independent experiment and gathers their respective signal. One example is a group of market research firms. Each firm conducts their own survey about the features of a specific market independent of other surveys. Generally, the accuracy of their signal about the state of the market depends heavily on the reliability and quality of their survey. Hence, signal accuracy can vary for each firm.

Example 3. (Conditionally Independent Information Structure) Table 3 shows the feature of a conditionally independent information structure for the case of three experts. The primitives of our model focuses the information structure (S, F), i.e., the grand coalition of experts' vector of signals and its corresponding conditional distributions. However, it is easier to understand this information structure as the product of individual expert's information structure. As seen in the table 3(b), the conditional probability that expert 1's signal matches the state is 0.75. For experts 2 and 3, the conditional probabilities of their signal matching the state are 0.70 and 0.65, respectively. We can see in column 2 of table 3 that the conditional probability that all of their signals match the state is equal to the product of their respective conditional probabilities of matching the state.

2.2.3 Monotonic Information Structures

Definition 4 (Monotonic Information Structure)**.** *An information structure* (S, F) *is called monotonic if it satisfies the monotone likelihood ratio property. That is, if for any pair s and* s' *in S where* $\#\{i | s(i) = a\} >$

	aaa	*aab*	*aba*	*baa*
A	$0.75 \times 0.70 \times 0.65$	$0.75 \times 0.70 \times 0.35$	$0.75 \times 0.30 \times 0.65$	$0.25 \times 0.70 \times 0.65$
B	$0.25 \times 0.30 \times 0.35$	$0.25 \times 0.30 \times 0.65$	$0.25 \times 0.70 \times 0.35$	$0.75 \times 0.30 \times 0.35$
	abb	*bab*	*bba*	*bbb*
A	$0.75 \times 0.30 \times 0.35$	$0.25 \times 0.70 \times 0.35$	$0.25 \times 0.30 \times 0.65$	$0.25 \times 0.30 \times 0.35$
B	$0.25 \times 0.70 \times 0.65$	$0.75 \times 0.30 \times 0.65$	$0.75 \times 0.70 \times 0.35$	$0.75 \times 0.70 \times 0.65$

(a) 3-person Conditionally Independent Information Structure

	a	*b*
A	0.75	0.25
B	0.25	0.75

(b) Expert 1's Information Structure

Table 3: Conditionally Independent Information Structure

$\#\{i|s'(i) = a\}$ *we have*

$$\frac{F_A(s)}{F_B(s)} > \frac{F_A(s')}{F_B(s')}.$$

A monotonic information structure is a generalization of a conditionally independent information structure. The same feature it shares is that more a signals in a signal vector indicates a better posterior belief for state A, and, analogously, more b indicate a higher posterior belief for B. It differs from conditional independence as the conditional probability of a vector of signal is not necessarily equal to the the product of conditional probability over the signals in that vector. Such generalization allows for the situation where experts can have correlated experiments. In the context of market research firms, surveys may be taken from the same pool of subjects. Hence, this induces positive correlation between the signal realizations of the experts.

Example 4. (Monotonic Information Structure) Table 3 shows the feature of a conditionally independent information structure for the case of three experts, which satisfies monotone likelihood ratio property. Consider signal vectors aaa and aab and their respective probabilities. Checking the condition for monotone likelihood ratio property, we get the following inequality $\frac{0.75\times0.70\times0.65}{0.25\times0.30\times0.35} > \frac{0.75\times0.70\times0.35}{0.25\times0.30\times0.65}$. This condition holds when comparing any two signals with one having at least one a signal more than the other.

2.2.4 Complementary Information Structures

Definition 5 (Complementary Information Structure)**.** *An information structure (S, F) is called complementary if it satisfies odd-odd or even-even property. That is, it satisfies odd-odd property if* $\dim(S) = 2^n$ *where n is odd and the support of $F_A(s)$ and $F_B(s)$ are $\{s|\#\{i|s(i) = a\}$ is odd$\}$ and $\{s|\#\{i|s(i) = a\}$ is even$\}$, respectively. It is said to satisfy even-even property if* $\dim(S) = 2^n$ *where n is even and the support of $F_A(s)$ and $F_B(s)$ are $\{s|\#\{i|s(i) = a\}$ is even$\}$ and $\{s|\#\{i|s(i) = a\}$ is odd$\}$, respectively.*

A complementary information structure says that we need every expert's signal to know the state of the world, which is then fully revealed. Anything that is less than the full vector of signals cannot help in determining the state of the world. One way to interpret this is the following scenario. Suppose that we have two experts with a common goal to know whether to launch a product that matches the taste of consumers in the market. Each expert is designated with answering a specific problem for a product launch. The first expert is tasked to know whether the market has a type A or type B consumers, which determines the taste of consumers. The second expert determines whether the product is a type A or type B, which indicates the compatibility of a product to the taste of consumers. If both experts get a signal that match, they recommend to launch the product in the market. If their signals disagree, then the recommendation is not to launch. However, notice that their individual signals cannot help in determining whether to launch a product or not. Such information structures entail strong complementarity among signals that neither one can be used to determine the state.

Example 5. (Complementary Information Structure) Table 4 shows the feature of a complementary information structure for the case of three experts. We can observe the support of the the conditional distribution F_A is the set of vector of signals with odd number of a signals $\{aaa, abb, bab, bba\}$. The complement set of signal vectors is the support set of F_B. The induced information structure for a single expert shows that it is uninformative. This is also the case for the two-expert coalition. Hence, any coalition of experts less than the grand coalition leads to an uninformative information structure.

	aaa	aab	aba	baa	abb	bab	bba	bbb
A	0.25	0	0	0	0.25	0.25	0.25	0
B	0	0.25	0.25	0.25	0	0	0	0.25

(a) 3-person Complementary Information Structure

	a	b
A	0.50	0.50
B	0.50	0.50

(b) Induced 1-person Information Structure

Table 4: Complementary Information Structure

2.3 Commission game

The game we introduce here uses the framework of cooperative game theory. The main advantage of this framework is to understand whether experts can rationally allocate the surplus they generate in an individual and coalitional sense. This is captured in the context of *transferable utility* games.

Definition 6 (Transferable Utility Game)**.** *A transferable utility game is a pair (N, v) where N is the set of players $N = \{1, 2, ..., n\}$ and $v : 2^N - \emptyset \rightarrow \mathbb{R}$ is a function that maps a coalition T to its worth $v(T)$.*

We refer to the transferable utility game that experts play as the commission game. To define the commission game, we need several parts and define the *worth of a coalition*. Fix a coalition of experts $T \subseteq N$ and consider a signal vector for that coalition s_T. Recall that the posterior belief for state A given signal s_T is $q(s_T) = \frac{F_A(s_T)}{F_A(s_T)+F_B(s_T)}$. We define the *interim* worth of coalition $v(T|s_T) = \max\{q(s_T), 1 - q(s_T)\} - \frac{1}{2}$. The $\frac{1}{2}$ that is subtracted from the maximum between the two posterior beliefs indicates the this interim worth of coalition is net of the worth of information that the investor has. In a sense, this is the expected value added to the investor for the signal s_T. We call the transferable utility game $(N, v|s)$ as the *interim commission game*. The worth of a coalition prior to observing the signals given state ω is $v(T|\omega) = \sum_{s_T} F(s_T|\omega)v(T|s_T)$. With this, we define the *ex ante* worth of coalition $v(T)$ given by

$$v(T) = \frac{1}{2}v(T|A) + \frac{1}{2}v(T|B).$$

The ex-ante worth of coalition T is the expected worth of a coalition of experts to the investor before they observe their signals. Similarly, we call the the transferable utility game (N, v) as the *ex-ante commission game*. Note that *ex-ante* commission game (N, v) is a weighted sum of *interim* commission games $(N, v|s)$. We show this in the following.

First, fix $T \subseteq N$. Then the *ex-ante* worth of coalition is given as the weighted average of *interim* worth of coalition. That is,

$$\begin{aligned}
v(T) &= \frac{1}{2}v_A(T) + \frac{1}{2}v_B(T) \\
&= \frac{1}{2}\sum_{s_T} F_A(s_T)v(T|s_T) + \frac{1}{2}\sum_{s_T} F_B(s_T)v(T|s_T) \\
&= \sum_{s_T} \left[F(s_T \cap A) + F(s_T \cap B)\right] v(T|s_T) \\
&= \sum_{s_T} F(s_T)v(T|s_T),
\end{aligned}$$

where $F(s_T \cap \omega) = \frac{0.5}{F_\omega(s_T)}$. Hence, we can rewrite the the *ex-ante* commission game (N, v) as follows.

Definition 7 (Commission Game)**.** *A TU game (N, v) is a commission game if there exists a triple (N, S, F) where N is the set experts, and (S, F) is an information structure such that*

$$v(T) = \left(\sum_{s_T \in S|_T} F(s_T)v(T|s_T)\right) = \left(\sum_{s_T \in S|_T} F(s_T)\max\{q(s_T), 1 - q(s_T)\}\right) - \frac{1}{2} \tag{1}$$

is satisfied for all $T \in 2^N$.

Example 2 continued. (Perfectly Correlated Information Structure) Refer to the Table 2 that shows a perfectly correlated information structure for the case of three experts. The corresponding *ex-ante* commission game is the following

$$v(T) = \begin{cases} 0.25 & \text{if } T \in 2^{\{123\}} - \emptyset \\ 0 & \text{otherwise} \end{cases}$$

We can see in this commission game with a perfectly correlated information structure that the worth of coalition is the same for any coalition of experts.

Example 3 continued. (Conditionally Independent Information Structure) Refer to the Table 3 that shows the feature of a conditionally independent information structure for the case of three experts. The corresponding *ex-ante* commission game is the following

$$v(T) = \begin{cases} 0.285 & \text{if } T = 123 \\ 0.25 & \text{if } T \in \{12, 13, 1\} \\ 0.20 & \text{if } T \in \{23, 2\} \\ 0.15 & \text{if } T = 3 \\ 0 & \text{otherwise} \end{cases}$$

Notice that there is no marginal improvement in the worth of a coalition whenever the signal accuracy of the incumbent coalition of experts is stronger than the entrant expert. Consider incumbent expert 1 and entrant expert 3. The marginal change in worth by adding expert 3 is zero. This non-improvement of the worth happens because expert 1 can overturn every signal expert 3 gets. That is, the recommendation of experts 1 and 3 always follows experts 1's signal regardless of the signal of expert 3. For the case where there is a positive marginal change, the should be a signal realization for the larger coalition that overturns the recommendation of a subset of that coalition. Consider the grand coalition of experts 123 and expert 1. If expert 1 receives signal a, then she recommends A. Suppose experts 2 and 3 both get signal b. On their own, they cannot change the recommendation of expert 1 but together they can overturn the recommendation of expert 1. Hence, abb is a signal that can overturn 1's recommendation, which gives a positive marginal change when combining experts' 2 and 3 with 1.

Example 4 continued. (Complementary Information Structure) Table 4 shows the feature of an information structure that satisfies odd-odd property for the case of three experts. The corresponding *ex-ante* commission game is the following

$$v(T) = \begin{cases} 0.50 & \text{if } T = 123 \\ 0 & \text{otherwise} \end{cases}$$

This commission game indicates that only the grand coalition of experts 123 has some worth. In particular, they reach the maximal worth while all other smaller coalitions reach the least possible worth. To see why this happens, we check the same principle of marginal change in worth for coalitions. Consider the signal bba. Expert 1's signal b cannot improve the posterior belief from the the prior. She recommends B but say that the investor's guess is as good as hers. Including expert 2's signal b cannot change expert 1's recommendation, so the marginal change in worth is zero. However, including all experts' bba changes this recommendation. Notice for any combined signal realization of the three experts that the recommendation of an immediate subset is overturned. Together, they recommend state A, which is reflected in the positive marginal change.

The main takeaway in commission games is the importance of overturning recommendations. For a coalition T and subcoalition $T - i$, the worth of coalition T will only be larger than the worth of $T - i$ if there is a signal vector for s_{T-i} such that $T - i$ experts' recommendation is overturned with the signal $s_T = s_{T-i} \times s_i$. If there is no signal that overturns the recommendation, then two coalitions T and T_i are equal. Now that we have constructed the game that summarizes the surplus experts generate, we move on the the solution concept that allocates the surplus among experts.

2.4 Core of commission game

The section above discussed the relevant game among expert that describes the surplus the can generate by agreeing to combine their information. Given this, each expert and every possible coalition of experts have a decision to make, i.e., whether or not they should consolidate their information or not. Their decision then depends on how much they are being compensated for joining the coalition compared to freelancing. Here, we introduce the *core* as the solution concept to allocate surplus generated by the grand coalition of experts.

Definition 8 (Core)**.** *The Core of a TU game (N, v) is the set $C(N, v) \subset \mathbb{R}^n$ such that $y \in C(N, v)$ satisfies*

1) *Experts' rationality:* $\sum_{i\in T} y(i) \geq v(T), \forall T \in 2^N - \emptyset$

2) *Efficiency:* $\sum_{i\in N} y(i) = v(N)$

This follows the classical definition of core allocation for non-zero sum game due to Gillies (1959). *Experts' rationality* is a combination of two forms of rationality– for individuals and for coalitions. Set the coalition T to be a singleton i. Experts' rationality says that $y(i) \geq v(T)$, meaning that a core allocation should give at least the amount that expert i can generate on its own. For a larger coalition T, the sum of core allocation for members of T should be weakly greater than the surplus they can generate together. Altogether these two capture rational cooperative behavior. The *efficiency* condition of the core essentially says that nothing is wasted as all the surpluses from the grand coalition of N is distributed among its members.

In the context of the commission game, a core allocation is an agreement about how much each expert gets if they all join up. That is, the core is implicitly assuming that the every expert needs to agree. Hence, we refer to this as *consolidating* their information where every expert agrees to make the grand coalition of experts. If one expert or a subset of experts disagree, then we assume that no subcoalition forms. We refer to this as *freelancing*. When answering whether agreement will happen among experts, we consider the ex-ante perspective such that an allocation is agreed upon prior to receiving their signals. Notice that if an agreement is not reached in the ex-ante commission game, then there is an interim commission game such that agreement cannot be reached as well. We show this below.

Proposition 1. *If the core exists for all interim commission game $(N, v|s)$, then the core exists for the ex-ante commission game.*

Proof. To prove this, we show the weighted sum of core allocation for *interim* commission games is a core allocation for the *ex-ante* commission game. First we show *expert's rationality* is satisfied. Fix $T \subseteq N$. Let $y(i|s_T)$ denote a core allocation for expert i in the interim commission game $(N, v|s)$ and define $y(i) \equiv \sum_{s_T\in S|_T} F(s_T)y(i|s_T)$. Since $y(i|s_T)$ is a core allocation, we have $\sum_{i\in T} y(i|s_T) \geq v(T|s_T)$. Multiplying both sides with the probability of a signal s_T, $F(s_T)$, and summing up across all signal realizations for coalition T, we get

$$\begin{aligned}
\sum_{s_T \in S|_T} F(s_T) \sum_{i \in T} y(i|s_T) &\geq \sum_{s_T \in S|_T} F(s_T) v(T|s_T) \\
\sum_{i \in T} \sum_{s_T \in S|_T} F(s_T) y(i|s_T) &\geq \sum_{s_T \in S|_T} F(s_T) v(T|s_T) \\
\sum_{i \in T} y(i) &\geq \sum_{s_T \in S|_T} F(s_T) v(T|s_T) \\
\sum_{i \in T} y(i) &\geq v(T).
\end{aligned}$$

The third line follows from definition of $y(i)$; and the fourth line from the *ex-ante* worth of coalition $v(N)$ is given as the weighted average of *interim* worth of coalitions $v(N|s)$.

We show that $y(i)$ satisfies *efficiency*. Taking the sum $y(i)$ for $i \in N$, we get

$$\begin{aligned}
\sum_{i \in N} y(i) &= \sum_{i \in N} \sum_{s \in S} F(s) y(i|s) \\
&= \sum_{s \in S} F(s) \sum_{i \in N} y(i|s) \\
&= \sum_{s \in S} v(N|s) \\
&= v(N),
\end{aligned}$$

where the third equality follows from the fact that $y(i|s)$ is core allocation for the interim commission game, and the four comes from again from the fact that the *ex-ante* worth of coalition $v(N)$ is equal to the weighted average of *interim* worth of coalitions $v(N|s)$. □

3 Results

3.1 Freelancing

We consider three standard information structures, i.e., *perfectly correlation*, *conditional independence*, and *monotonic* information structures. The results below show that commission games with these forms of of information structure have an empty core. That is, the grand coalition of experts cannot agree upon an allocation that satisfies experts' rationality and efficiency. The symmetry we impose on the information structures implies that the worth of a coalition only depends on the size of the coalition fixing the size of the grand coalition, Lastly, we show that conditional independence while relaxing symmetry also give the same results.

The main approach in proving emptiness of the core is the following necessary condition. The core of (N, v) is non-empty only if for any partition of N $\{T_1, T_2, ..., T_l\}$ we have

$$v(T_1) + v(T_2) + ...v(T_l) \leq v(N).$$

Our first result on the emptiness of core says that if the commission game has a perfectly correlated information structure, then experts would not consolidate their information but rather freelance.

Proposition 2. *Let a commission game (N, v) have an associated information structure (S, F). If (S, F) is a perfectly correlated information structure, then the core is empty for $|N| = n \geq 2$.*

Proof. Fix $n \geq 2$. Let $F_A(a...a) = F_B(b...b) = \gamma > 0.5$ and suppose core exists. Then, a necessary condition of the core says

$$\begin{aligned} v(N-i) + v(i) &\leq v(N) \\ \gamma - 0.5 + \gamma - 0.5 &\leq \gamma - 0.5 \\ \Rightarrow \; 2(\gamma - 0.5) &\leq \gamma - 0.5. \end{aligned}$$

Since $\gamma > 0.5$, then the inequality implies $2 \leq 1$. This is a contradiction. □

Remark 1. *On a more general perfectly correlated information structure.* Note that proposition still holds if we weaken the anonymity assumption in the following sense. Let $F_A(a...a) = \gamma_a$ and $F_B(b...b) = \gamma_b$ where $\gamma_a, \gamma_b > 0.5$. Define $\hat{\gamma} = 0.5\gamma_a + 0.5\gamma_b$. The proposition still holds if you replace γ with $\hat{\gamma}$.

The second result on the emptiness of core says that if the commission game has a conditionally independent information structure, then experts would not consolidate their information but rather freelance.

Proposition 3. *Let a commission game (N, v) have an associated information structure (S, F). If (S, F) is a conditionally independent information structure, then the core is empty for $|N| = n \geq 2$.*

Proof. Consider (N, v) with a conditionally independent information structure (S, F). First, note that v is weakly increasing. That is, for each $T \subseteq N$,

$$v(T \cup i) \geq v(T), \forall i \in N.$$

This follows from the assumption that signals are conditionally independent and informative. Second, if the strict inequality holds, then there exists a pair of signals $s_{T \cup i}$ and s_T such that recommendation σ is chosen given $s_{T \cup i}$ while recommendation $\neg\sigma$ is chosen given s_T. To see this, suppose that the implication is not true. If there is not such a pair, then the introduction of i's signal does not change the interim decisions of the coalition T. Hence, for any $s_T \in \times_{j \in T}(s_j)$, we have

$$v(T \cup i | s_{T \cup i}, s(i) = a) + v(T \cup i | s_{T \cup i}, s(i) = b) = v(T | s_T).$$

Summing this up across s_T, we get that $v(T \cup i) = v(T)$, a contradiction.

Now, we show that core is empty. A necessary condition for core to exist is that

$$v(N) - v(N - i) \geq v(i), \forall i \in N.$$

Let γ_i denote $F_A(a_i) = F_B(b_i)$. Then, $v(i) = \gamma - \frac{1}{2} > 0$. As v is weakly increasing, we have two cases. For the first case, we have $v(N) - v(N - i) = 0$. This necessary condition trivially fails to hold as $v(i) > 0$.

Consider the second case where $v(N) - v(N - i) > 0$. Let S^A_{N-i} and S^B_{N-i} denote the sets of signal vectors in $S|_{N-i}$ where coalition $N - i$ recommends A and B, respectively. Now, we expand the term $v(N)$.

$$
\begin{aligned}
v(N) - \frac{1}{2} &= \sum_{s \in S_N} (\max\{F(A)F_A(s), F(B)F_B(s)\}) \\
&= \frac{1}{2} \sum_{s \in S_N} (\max\{F_A(s), F_B(s)\}) \\
&= \frac{1}{2} \sum_{s' \in S_{N-i}} \Big(\max\{F_A(s = s' \times a), F_B(s = s' \times a)\} + \max\{F_A(s = s' \times b), F_B(s = s' \times b)\} \Big) \\
&= \frac{1}{2} \sum_{s' \in S^A_{N-i}} \Big(\max\{F_A(s = s' \times a), F_B(s = s' \times a)\} + \max\{F_A(s = s' \times b), F_B(s = s' \times b)\} \Big) \\
&+ \frac{1}{2} \sum_{s' \in S^B_{N-i}} \Big(\max\{F_A(s = s' \times a), F_B(s = s' \times a)\} + \max\{F_A(s = s' \times b), F_B(s = s' \times b)\} \Big) \\
&= \frac{1}{2} \sum_{s' \in S^A_{N-i}} (\max\{F_A(s = s' \times a), F_B(s = s' \times a)\}) + \frac{1}{2} \sum_{s' \in S^A_{N-i}} (\max\{F_A(s = s' \times b), F_B(s = s' \times b)\}) \\
&+ \frac{1}{2} \sum_{s' \in S^B_{N-i}} (\max\{F_A(s = s' \times a), F_B(s = s' \times a)\}) + \frac{1}{2} \sum_{s' \in S^B_{N-i}} (\max\{F_A(s = s' \times b), F_B(s = s' \times b)\}) \\
&= \frac{1}{2} \sum_{s' \in S^A_{N-i}} F_A(s = s' \times a) + \frac{1}{2} \sum_{s' \in S^A_{N-i}} (\max\{F_A(s = s' \times b), F_B(s = s' \times b)\}) \\
&+ \frac{1}{2} \sum_{s' \in S^B_{N-i}} (\max\{F_A(s = s' \times a), F_B(s = s' \times a)\}) + \frac{1}{2} \sum_{s' \in S^B_{N-i}} F_B(s = s' \times b) \\
&= \sum_{s' \in S^A_{N-i}} F_A(s = s' \times a) + \sum_{s' \in S^A_{N-i}} (\max\{F_A(s = s' \times b), F_B(s = s' \times b)\}),
\end{aligned}
$$

where the second line follows from conditional independence; the fourth equality follows from separating the signal vectors according to S^A_{N-i} and S^B_{N-i}; second to the last equality follows from the fact that for any $s' \in S^A_{N-i}$, $\max\{F_A(s = s' \times a), F_B(s = s' \times a)\} = F_A(s = s' \times a)$; and the last equality follows from $F_\omega(s) = F_{\neg\omega}(\neg s)$.

Similarly, we expand the term $v(N-i)$.

$$
\begin{aligned}
v(N-i) - \frac{1}{2} &= \sum_{s' \in S_{N-i}} (\max\{P(A)P(s'|A), P(B)P(s'|B)\}) \\
&= \frac{1}{2} \sum_{s' \in S^A_{N-i}} F_A(s') + \frac{1}{2} \sum_{s' \in S^B_{N-i}} F_B(s') \\
&= \frac{1}{2} \sum_{s' \in S^A_{N-i}} F_A(s')F_A(s_i = a) + \frac{1}{2} \sum_{s' \in S^A_{N-i}} F_A(s')F_A(s_i = b) \\
&+ \frac{1}{2} \sum_{s' \in S^B_{N-i}} F_B(s')F_B(s_i = a) + \frac{1}{2} \sum_{s' \in S^B_{N-i}} F_B(s')F_B(s_i = b) \\
&= \sum_{s' \in S^A_{N-i}} F_A(s')F_A(s_i = a) + \sum_{s' \in S^A_{N-i}} F_A(s')F_A(s_i = b),
\end{aligned}
$$

where the equalities follow from the same argument as in the $v(N)$ expansion. Let the $\hat{S}_{N-i}$ denote the set of all s_{N-i} where A is recommended but switched to B for the signal $s_N = s_{N-i} \times b$.

$$
\begin{aligned}
v(N) - v(N-i) &= \sum_{s' \in S^A_{N-i}} (\max\{F_A(s = s' \times b), F_B(s = s' \times b)\} - F_A(s')F_A(s_i = b)) \\
&= \sum_{s' \in \hat{S}_{N-i}} (F_B(s = s' \times b) - F_A(s')F_A(s_i = b)) \\
&= \sum_{s' \in \hat{S}_{N-i}} F_B(s')F_B(s_i = b) - F_A(s')F_A(s_i = b) \\
&= \gamma_i \left(\sum_{s' \in \hat{S}_{N-i}} F_B(s') \right) - (1 - \gamma_i) \left(\sum_{s' \in \hat{S}_{N-i}} F_A(s') \right),
\end{aligned}
$$

where the second equality follows from keeping the nonzero term; third equality from conditional independence; and last equality from $F_A(s_i = b) = 1 - \gamma_i$ and $F_B(s_i = b) = \gamma_i$. Notice that for any $s' \in \hat{S}_{N-i}$ we have $F_B(s') \leq F_A(s')$. So, $\left(\sum_{s' \in \hat{S}_{N-i}} F_B(s') \right) \leq \left(\sum_{s' \in \hat{S}_{N-i}} F_A(s') \right)$. Denote the left-hand side and right-hand side of this inequality by L and H, respectively. Manipulating the inequality of the necessary condition, we get

$$
\begin{aligned}
& v(N) - v(N-i) \geq v(i) \\
\implies & \gamma_i L - (1-\gamma_i)H \geq \gamma_i - \frac{1}{2} \\
\implies & \gamma_i(L+H-1) \geq H - \frac{1}{2} \\
\implies & \gamma_i \leq \frac{\frac{1}{2}-H}{1-(L+H)} \\
\implies & \gamma_i \leq \frac{\frac{1}{2}-H}{(\frac{1}{2}-H)+(\frac{1}{2}-L)} \\
\implies & \gamma_i \leq \frac{1}{2},
\end{aligned}
$$

where fourth line follows from $\left(\sum_{s' \in \hat{S}_{N-i}} F_B(s')\right) + \left(\sum_{s' \in \hat{S}_{N-i}} F_A(s')\right) \leq 1$; and last line follows from H is greater than L. However, this result contradicts our assumption that for any i, $\gamma_i > \frac{1}{2}$. This completes the proof. □

Remark 2. *On anonymous information structure*. A perfectly correlated information structure implies that it is anonymous. If we impose this on the conditional independence result, we simply get standard assumption that all agents are drawing the identical distribution.

Remark 3. *On subcoalition formation*. Notice that core assumes that the grand coalition of N. If we restrict the information structure to a coalition T then that information structure is still conditionally independent. This means the proposition holds for this restricted coalition implying that a smaller coalition of experts cannot consolidate their information as well.

The final result on the emptiness of core says that if the commission game has a monotonic information structure, then experts would not consolidate their information but rather freelance. We prove it in two parts where we separate odd and even case. To do so, we introduce Lemma 1 for the case where the size of the grand coalition is odd.

Lemma 1. *Let the (S,F) be an anonymous information structure of the commission game (N,v). Further-*

more, let the information structure satisfy

$$F_a(\#a = k) = F_b(\#b = k) = \gamma_k,$$

and for any $k \geq \frac{n+1}{2}$, $\gamma_k \geq \gamma_{n-k}$. *Let* $n \geq 3$ *be odd. Then*

$$v(N) - \sum_{i \in N} v(i) \leq \frac{n-1}{2}(\gamma_0 - \gamma_n)$$

Proof. Let (S, F) be an anonymous information structures. Then, it follows that $v(T) = v(T')$ for any $|T| = |T'|$. Furthermore, it can be verified that the worth of the grand coalition of expert is given by

$$v(N) = \sum_{k=\frac{n+1}{2}}^{n} \binom{n}{k} \gamma_k - \frac{1}{2},$$

and the worth of a single expert is

$$v(i) = \sum_{k=0}^{n-1} \binom{n-1}{k} \gamma_{k+1} - \frac{1}{2}.$$

Note that

$$
\begin{aligned}
v(N) - \sum_{i \in N} v(i) &= v(N) - n \cdot v(i) \\
&= \sum_{k=\frac{n+1}{2}}^{n} \binom{n}{k} \gamma_k - \frac{1}{2} - \sum_{m=0}^{n-1} n \binom{n-1}{m} \gamma_{m+1} + \frac{n}{2} \\
&= \frac{n-1}{2} + \sum_{k=\frac{n+1}{2}}^{n} \binom{n}{k} \gamma_k - \sum_{m=0}^{n-1} n \binom{n-1}{m} \gamma_{m+1} \\
&= \frac{n-1}{2} - (n-1)\gamma_n - \left\{ \sum_{k=\frac{n+1}{2}}^{n-1} \left[n \binom{n-1}{k-1} - \binom{n}{k} \right] \gamma_k \right\} - \sum_{m=0}^{\frac{n-3}{2}} n \binom{n-1}{m} \gamma_{m+1} \\
&= \frac{n-1}{2} - (n-1)\gamma_n - \sum_{k=\frac{n+1}{2}}^{n-1} \binom{n}{k} (k-1) \gamma_k - \sum_{m=0}^{\frac{n-3}{2}} n \binom{n-1}{m} \gamma_{m+1} \\
&= \frac{n-1}{2} - (n-1)\gamma_n - \sum_{k=\frac{n+1}{2}}^{n-1} \binom{n}{k} (k-1) \gamma_k - \sum_{m=1}^{\frac{n-1}{2}} n \binom{n-1}{m-1} \gamma_m \\
&= \frac{n-1}{2} - (n-1)\gamma_n - \sum_{k=\frac{n+1}{2}}^{n-1} \binom{n}{k} (k-1) \gamma_k - \sum_{m=1}^{\frac{n-1}{2}} m \binom{n}{m} \gamma_m
\end{aligned}
$$

where the fifth and last equality follow from the fact that $n\binom{n-1}{k-1} = k\binom{n}{k}$ and the second to last equality follows from making an appropriate change in the indices in the final summation. Recall that $\gamma_0 = 1 - \sum_{k=1}^{n} \gamma_k$. Adding and subtracting $\frac{n-1}{2}\gamma_0$ to the above yields

$$
\begin{aligned}
v(N) - n \cdot v(i) &= \frac{n-1}{2} - (n-1)\gamma_n - \sum_{k=\frac{n+1}{2}}^{n} \binom{n}{k} (k-1) \gamma_k - \sum_{m=1}^{\frac{n-1}{2}} m \binom{n}{m} \gamma_m + \frac{n-1}{2}\gamma_0 - \frac{n-1}{2}\gamma_0 \\
&= \frac{n-1}{2} - (n-1)\gamma_n - \sum_{k=\frac{n+1}{2}}^{n-1} \binom{n}{k} (k-1) \gamma_k - \sum_{m=1}^{\frac{n-1}{2}} m \binom{n}{m} \gamma_m + \frac{n-1}{2}\gamma_0 - \frac{n-1}{2} \left[1 - \sum_{l=1}^{n} \binom{n}{l} \gamma_l \right] \\
&= \frac{n-1}{2}(\gamma_0 - \gamma_n) + \frac{n-1}{2} \sum_{l=1}^{n-1} \binom{n}{l} \gamma_l - \sum_{k=\frac{n+1}{2}}^{n-1} \binom{n}{k} (k-1) \gamma_k - \sum_{m=1}^{\frac{n-1}{2}} m \binom{n}{m} \gamma_m
\end{aligned}
$$

Collating the indices together yields:

$$v(N) - n \cdot v(i) = \frac{n-1}{2}(\gamma_0 - \gamma_n) - \left[\sum_{k=\frac{n+1}{2}}^{n-1} \binom{n}{k}\left(k - 1 - \frac{n-1}{2}\right)\gamma_k\right] + \left[\sum_{m=1}^{\frac{n-1}{2}} \binom{n}{m}\left(\frac{n-1}{2} - m\right)\gamma_m\right]$$

Since $\gamma_k \geq \gamma_{n-k}$ for all $k \geq \frac{n+1}{2}$, we have

$$\begin{aligned} v(N) - n \cdot v(i) &\leq \frac{n-1}{2}(\gamma_0 - \gamma_n) - \left[\sum_{k=\frac{n+1}{2}}^{n-1} \binom{n}{k}\left(k - 1 - \frac{n-1}{2}\right)\gamma_{n-k}\right] + \left[\sum_{m=1}^{\frac{n-1}{2}} \binom{n}{m}\left(\frac{n-1}{2} - m\right)\gamma_m\right] \\ &= \frac{n-1}{2}(\gamma_0 - \gamma_n) \end{aligned}$$

where the last step follows from the fact that $\binom{n}{k} = \binom{n}{n-k}$ and expansion and cancellation of all terms in the summations. □

Proposition 4. *Let a commission game (N, v) have an associated information structure (S, F). If (S, F) is a monotonic and anonymous information structure, then the core is empty for $|N| = n \geq 2$.*

Proof. Suppose that the core exists. Then there exists a $y \in \mathbb{R}^n$ such that

$$v(N) = \sum_{i \in N} y(i) \text{ and for all } i \; y(i) \geq v(i)$$

Case 1: *Odd n.* Suppose, without loss of generality that for some $i \in N$, $y(i) > v(i)$. Then it must be the case that $v(N) = \sum_{i \in N} y(i) > \sum_{i \in N} v(i)$. Notice that a monotonic and anonymous information structure satisfies the sufficient condition in Lemma 1. Therefore we have $v(N) \leq \sum_{i \in N} v(i)$ by Lemma 1, which leads to a contradiction.

Case 2: *Even n.* It can be verified that the worth of the grand coalition is the following

$$v(N) = \frac{1}{2}\binom{n}{\frac{n}{2}}\gamma_{\frac{n}{2}} + \sum_{k=\frac{n}{2}+1}^{n} \binom{n}{k}\gamma_k - \frac{1}{2}$$

Furthermore, for any $i \in N$ we have

$$v(N-i) = \sum_{k=\frac{n}{2}}^{n-1} \binom{n-1}{k}(\gamma_{k+1}+\gamma_k) - \frac{1}{2} = \sum_{k=\frac{n}{2}+1}^{n} \left(\binom{n-1}{k} + \binom{n-1}{k-1}\right)\gamma_k + \binom{n-1}{\frac{n}{2}}\gamma_{\frac{n}{2}} - \frac{1}{2} = v(N).$$

Hence, the left-hand side of the necessary condition is 0. Furthermore, the worth of any singleton coalition is given by

$$v(i) = \sum_{k=0}^{n-1} \binom{n-1}{k}\gamma_{k+1} - \frac{1}{2} \geq 0.$$

Note that this holds with strict inequality as $\gamma_k > \gamma_{n-k}$ for any $k \geq \frac{n+1}{2}$. Therefore, the necessary condition does not hold, which completes the proof. □

Remark 4. *On subcoalition formation with monotonic information structure* It is immediate from the proposition above that restricting the information structure to a coalition T will have a commission game that has an empty core. This is from the fact that the monotonic information structure restricted to T is still monotonic. This implies a smaller coalition of experts cannot consolidate their information as well.

Remark 5. *On perturbations of information structure* The propositions on emptiness of the core are robust to small changes in the information structure. First, the game is continuous in F. For fix N, the game is v is a mapping from the space of (S, F) $\mathbb{R}^{2^n}$ to space of games $\mathbb{R}^{n-1}$. Note that the commission game is a a sum of the maximum of posteriors. The posteriors $q(s_T)$ are continuous in $F(s_T)$ and the maximum operator is continuous given that the posteriors are continuous. Hence the game, which is a sum of continuous, is continuous. Second, notice that the game (N, v) is in the space $\mathbb{R}^{n-1}$. The set of games that have a core is closed. Hence, the set of games with empty core is open. Therefore, from continuity of games and openess of the set of games with empty core, it follows that for a sufficiently small ε distance from a perfectly correlated, conditionally independent, or monotonic (S, F) the core of its corresponding perturbed commission is also empty.

3.2 Consolidating

The previous section enumerated three cases that lead to freelancing. Here, we present two results on the existence of a core allocation. First, we show a condition on the information structure where the necessary condition of the core to exist means that there is a unique core allocation that is exactly the individual worth of each expert.

Proposition 5. *Let the (S, F) be an anonymous information structure of the commission game (N, v). Furthermore, let the information structure satisfy*

$$F_a(\#a = k) = F_b(\#b = k) = \gamma_k,$$

and for any $k \geq \frac{n+1}{2}$, $\gamma_k \geq \gamma_{n-k}$. Let $n \geq 3$ be odd. If the Core exists, then for each $i \in N$ we have $y(i) = v(i)$.

Proof. Suppose that the core exists. Then there exists a $y \in \mathbb{R}^n$ such that

$$v(N) = \sum_{i \in N} y(i) \text{ and for all } i \; y(i) \geq v(i)$$

Suppose, without loss of generality that for some $i \in N$, $y(i) > v(i)$. Then it must be the case that $v(N) = \sum_{i \in N} y(i) > \sum_{i \in N} v(i)$. However, by Lemma 1, we have that $v(N) \leq \sum_{i \in N} v(i)$ which is a contradiction. □

In our second result, we provide a sufficient condition, i.e., complementary information structure, that gives a non-empty core. To do so, we present a result that shows that the worth of a coalition smaller than the grand coalition is zero for such information structure.

Lemma 2. *If the information structure is complementary and uniform over the support, then the worth of any coalition smaller than the grand coalition of experts is zero.*

Proof. We consider the odd-odd case (the even-even case is analagous). Consider n odd and suppose the information structure is odd-odd with a uniform distribution over the support. Let S_A be the set of signals in which the number of a signals is odd (likewise the number of b signals is even). Furthermore, let S_B be the set of signals with an odd number of b signals (likewise an even number of a signals). Given that $F_A(s)$ and $F_B(s)$ have uniform distributions over S_A and S_B respectively, it is easy to see that for any given $s \in S_A$, $F_A(s) = \frac{1}{2^{n-1}}$ and $F_B(s) = 0$. Likewise, for any $s \in S_B$, $F_B(s) = \frac{1}{2^{n-1}}$ and $F_A(s) = 0$. It is clear that the information structure (S, F) is fully revealing and therefore, $v(N) = \frac{1}{2}$.

We now show that for any sub-coalition $T \subset N$, $v(T) = 0$. We do so using recursion. Consider first a coalition $N - 1$.[2] Consider an arbitrary signal $s \in S_{N-1}$. Then $F_A(s) = F_A(s \times a) + F_A(s \times b)$ where $s \times x \in S$ is a signal profile from the information structure (S, F). Given that (S, F) is uniform and odd-odd, it is easy to show that $\Pr(s|A) = \frac{1}{2^{n-1}}$.[3] Similarly, we can show that $F_B(s) = \frac{1}{2^{n-1}}$. Therefore, $F_A(s) = F_B(s) = \frac{1}{2^{n-1}}$ for any $s \in S_{N-1}$. Clearly, S_{N-1} is uninformative and therefore $v(N-1) = 0$.

Consider now a coalition $N - 2 \subset N - 1 \subset N$. Consider an arbitrary signal profile $s \in S_{N-2}$. Then $F_A(s) = F_A(s \times a) + F_A(s \times b)$ where $s \times x \in S_{N-1}$ is a signal profile from S_{N-1}. Given the above, it is clear that $F_A(s) = \frac{1}{2^{n-2}}$. Similarly, $F_B(s) = \frac{1}{2^{n-2}}$. Given this is an arbitrary signal profile, it is clear that S_{N-2} is uninformative and therefore $v(T - 2) = 0$.

Recursively, it follows that for any $k \in 1, 2, \ldots, n - 1$, for any signal profile $s \in S_{N-k}$, $F_A(s) = F_B(s) = \frac{1}{2^{n-k}}$ and therefore S_{N-k} is uninformative. Hence, $v(N - k) = 0$ for any $k \in 1, 2, \ldots, n - 1$. Since $N - k$ is anonymous to any composition of coalition members, it is clear that $v(T) = 0$ for any $T \subset N$. □

Proposition 6. *If the associated information structure (S, F) of the commission game (N, v) is complementary and uniform over the support, then the core of the commission game is non-empty.*

Proof. Choose any $y = (y(1), ..., y(n))$ such that for any $i \in N$ $y(i) \geq 0$ and $\sum_{i \in N} y(i) = v(N)$. Fix $T \subset N$. From the lemma above, $v(T) = 0$ for any $T \subset N$. Therefore, $\sum_{i \in T} y(i) \geq v(T)$. □

Remark 6. *On perturbations.* Note that the existence of core for a commission game with complementary information structure is robust to small changes. Consider an example from the table below. The commission game association to this information structure is $V(12) = 0.5 - \varepsilon$ and

[2]For brevity, we consider a coalition $N - k$ to denote an arbitrary coalition of size $n - k$.

[3]To see this, suppose the number of a signals in s is even. Then (s, a) has an odd number of a signals and (s, b) has an even number of b signals. Given odd-odd and uniform, $F_A(s) = \frac{1}{2^{n-1}} + 0$. The case for odd a signals in s is analogous.

$v(1) = v(2) = 0$. Any nonnegative allocation $y(1)$ and $y(2)$ such that $y(1) + y(2) = 0.5 - \epsilon$ are core allocations so long as ε is small.

	aa	ab	ba	bb
A	$0.5 - \varepsilon$	ε	ε	$0.5 - \varepsilon$
B	ε	$0.5 - \varepsilon$	$0.5 - \varepsilon$	ε

(a) 2-person ε-complementary Information Structure

	a	b
A	0.5	0.5
B	0.5	0.5

(b) Induced 1-person Information Structure

Table 5: ε-complementary information structure

4 Discussion

The remarks on the freelancing cases indicate that not only consolidation cannot form for the grand coalition but for every smaller coalition than that. An analogous, although not obvious, statement can be said for the complementary information structure. By restricting the information to a specific coalition T, we get a game where no expert from T has an informative signal. Trivially, the core exists where every expert gets nothing. Hence, a smaller group of experts can potentially consolidate. This insight on freelancing and consolidating for smaller coalitions may generalize to be true for other information structure not discussed here.

One assumption that possibly drives the negative result is that there are no economies of scale, i.e., the coalition of experts can only cater to at most one client. If we can allow for coalitions of experts to increase their capacity along the size of their coalition, then we can allow for cooperation to exist. To see this, suppose that a T-sized coalition formed to cater to *Investor A*. If can the coalition of experts can costlessly reproduce the information for *Investor B*, they can charge as much from B as they did from A. Hence, a sort of linear economies of scale can trivially result in consolidation of experts. This is true to conditionally independent signals. However, this seems to not be the case for perfectly correlated signal as the experts cannot learn more about the state of the world from banding together. Now, we can see that the results above may come from a special knife-edge case where cost or reproducing the information is infinitely high. If we can allow for a linear cost of reproduction for every new client up to capacity, then we can get a cutoff level for cost where above it experts choose to freelance and below to consolidate. Possibly, this can order the set of information structures according to this indifference rule.

Another line of research one can explore is the complementary information structure. Our result shows that the complementary structure yields a large core as the worth of the sub-coalitions is exactly zero. This means that the core includes the allocations at the extreme with just one expert getting all the surplus or the middle where everyone gets an equal split or anything in between. In a sense, the core prescribes too much allocations that it is no longer meaningful to use. One direction we can take is to weaken the notion of complementary information structure. Consider table 6.

	aa	ab	ba	bb
A	0.5	ε	ε	$0.5 - 2\varepsilon$
B	ε	$0.5 - \varepsilon$	$0.5 - \varepsilon$	ε

(a) 2-person weak complementary Information Structure

	a	b
A	$0.5 + \varepsilon$	$0.5 - \varepsilon$
B	0.5	0.5

(b) Induced 1-person Information Structure

Table 6: Weak complementary information structure

The worth of the 2-person grand coalition and 1-person coalition are equal to $0.5 - 2\varepsilon$ and $\frac{\varepsilon}{2}$, respectively, given the epsilon is small. Notice that this weakening lowers the worth of the grand coalition while increasing the worth of the singleton coalition. Furthermore, the equal split allocation of $0.25 - \varepsilon$ is still in the core so long as ε is less than 0.167.

Lastly, we note one thing about weakening the complementary result. There is a possible limiting case where experts are exactly indifferent between consolidating and freelancing. We call such limiting information structure as *majority correlated*, which we show by example in table 7. The unique core allocation for this commission game is $(\frac{1}{6}, \frac{1}{6}, \frac{1}{6})$, which is exactly the worth of a single expert. This is another direction we can take in better understanding the general characterization of the core in commission games.

	aaa	aab	aba	baa	abb	bab	bba	bbb
A	0	0.33	0.33	0.33	0	0	0	0
B	0	0	0	0	0.33	0.33	0.33	0

(a) 3-person Majority Correlated Information Structure

	a	b
A	0.666	0.333
B	0.333	0.666

(b) 1-person Information Structure

Table 7: Majority Correlated Information Structure

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